Higher Order Parabolic Approximations for Sound Propagation in Stratified Moving Media

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Asymptotic solutions of order k^{-n} are developed for the equations governing sound propagation through a stratified moving medium. Here k is a dimensionless wave number and n the arbitrary order of the approximation. These approximations are an extension of geometric acoustics theory and provide corrections to that theory in the form of multiplicative functions that satisfy parabolic partial differential equations. These corrections account for the diffraction effects caused by variation of the field normal to the ray path and the interaction of these transverse variations with thevariationof the field along the ray. The theory is illustrated by application to simple examples.

Introduction

THE use of a parabolic equation to obtain approximate solutions of the elliptic equation governing the propagation of time-harmonic waves was pioneered by Leontovitch and Fock[1] in the context of long-distance tropospheric radiowave propagation and the diffraction of those waves by the spherical Earth. Since then the parabolic equation method (PEM) has been applied for the soution of problems of radiowave diffraction,[2] laser beam propagation,[3] and the propagation of acoustic disturbances through random media.[4] The most recent use of the method has been in the study of long-distance acoustic propagation in the ocean.[5] As generally derived, the parabolic approximation is valid only for propagation through stationary media with slowly varying ambient properties. It is further restricted in that no means of improving the approximation is provided. Nonetheless, for many applications, the PEM provides an attractive alternative to the use of the geometric acoustics theory, which is a rational approximation, since diffraction effect, unaccounted for in the lowest order geometric theory, are included in PEM solutions. However, for aeroacoustic analyses the limitation to stationary media is particularly restrictive.

Some of these restrictions have been removed in several relatively recent investigations. An analysis by Kriegsmann and Larsen[6] presents a new parabolic approximation based on the lowest order geometric acoustics theory. In that study attention was restricted to the reduced wave equation with a variable index of refraction; hence, mean motion of the medium was not considered. However, the theory developed in that analysis extended the applicability of parabolic approximations to include propagation in stationary media with $\mathcal{O}(1)$ refractive index variation.

Since a geometric acoustics approximation for wave propagation in moving media is well developed,[7] the work of Ref. 6 implies that a parabolic approximation for moving media could be based on that theory. For propagation in stratified moving media, a parabolic approximation has been so developed.[8,9] With the analyses of Refs. 6, 7, and 9 the restrictions of applying only to slowly varying stationary media have been removed from certain forms of parabolic approximations. It is important to note that the parabolic approximations developed from the geometric acoustics theory differ significantly from previous parabolic approximations. However, even these most recent analyses have not provided a means of improving the basic approximation.

It is the purpose of this paper to develop a sequence of successively more accurate parabolic approximations that describe the propagation of acoustic disturbances through a stratified moving medium. These approximations are based on the geometric acoustics theory, and may be considered an extension of that theory. In fact, the first member of the sequence is identical to the lowest order geometric acoustics theory, while the second is the same as the parabolic approximation as derived in Ref. 9. This first-order parabolic theory improves upon the geometric theory, here considered the zeroth-order parabolic theory, by containing diffraction effects neglected in the geometric theory. Each subsequent member of the sequence of approximations describes the diffraction effects with a higher degree of precision. In the next section, the sequence of approximations is developed for plane waves in stationary media. This simple example is used to display the analysis leading to the sequence of approximations in the simplest possible case. Subsequently, the analysis of acoustic disturbances propagating in a stratified moving medium is discussed. As this is the most general form of the theory to be considered herein, it is not entirely suited to exposition. Hence, merely the outline of the analysis is presented. For the purpose of clarifying the method, consideration is restricted to plane waves propagating in a sheared flow with a constant index of refraction. This case may be completely analyzed by Fourier analysis and numerical integration of the resulting ordinary differential equations. Hence, an additional benefit of this example is a comparison of the approximate solutions with the exact solutions, a comparison which is then presented.

Plane Waves in a Homogeneous Stationary Medium

In order to establish the general outline of the analysis to follow, and to illustrate the basic nature of the approximation to be obtained, consider solutions of the reduced wave equation

$$p_{xx}+p_{yy}+k^2p=0 \tag{1}$$

of the form

$$p=A(x;k)e^{iky} \tag{2}$$

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as the dimensionless wave number $k \to \infty$. Substitution of Eq. (2) into Eq. (1) yields

$$\frac{\partial A_y}{\partial y} + 2ikA_y = -A_{xx} \tag{3}$$

for the unknown function $A(x,y)$. Here, the mixed notation $\partial A_y/\partial y$ for the second derivative of A with respect to y is introduced since, in the following, Eq. (3) will be temporarily thought of as an inhomogeneous first-order ordinary differential equation for A_y.

In general, solutions of Eq. (1) would be sought which, for example, reduce to $p|_{y=0} = p_0(x)$ and satisfy a radiation condition as $y \to +\infty$. The form of the solution as given in Eq. (2) anticipates satisfaction of this second condition. [It should be noted that the $\exp(-i\omega t)$ convention is used.] If A_{xx} is assumed to be $\mathcal{O}(k^0)$ as $k \to \infty$, Eq. (3) suggests that A_y is either $\mathcal{O}(k)$ or $\mathcal{O}(1/k)$ in the given limit. The first of these possibilities implies that $A \sim \exp(-2iky)$ and would provide a solution that fails to satisfy the required radiation condition as $y \to +\infty$. The second possibility implies that A_{yy} is $\mathcal{O}(1/k)$ and that Eq. (3) may be approximated by

$$2ikA_y + A_{xx} = 0 \tag{4}$$

This is the usual form of the parabolic approximation for quasiplane waves. Similar approximations can be obtained for waves that are essentially of cylindrical or spherical geometry.[10] As derived, Eq. (4) provides an irrational approximation in that no method of improving the approximation is given. Essentially this is the customary derivation. Here a different approach is used. If Eq. (3) is multiplied by the integrating factor $\exp(2iky)$ and integrated with respect to y,

$$A_y = -\int_{i\infty}^{y} e^{2ik(\phi - y)} A_{xx} \mathrm{d}\phi \tag{5}$$

is obtained. Here it is assumed that A and all of its derivatives are finite as $y \to i\infty$. This condition ensures that terms proportional to $\exp(-2iky)$ and, therefore, incapable of satisfying the required radiation condition, do not occur in the final solution. Repeated integration of the right-hand side of Eq. (5) by parts yields

$$A_y \simeq \sum_{j=0}^{\infty} \left(\frac{i}{2k}\right)^{j+1} \frac{\partial^j A_{xx}}{\partial y^j} \tag{6}$$

Note that the function $A(x;k)$ must satisfy the auxiliary condition $A|_{y=0} = p_0(x)$. If terms of $\mathcal{O}(1/k)$ are neglected, Eq. (6) becomes $A_y = 0$, which implies that, to this order,

$$p = p_0(x)e^{iky} \tag{7}$$

which is what would be obtained if a simple, lowest order, geometric acoustics solution of Eq. (1) were sought. Maintaining $\mathcal{O}(1/k)$ terms in Eq. (6) yields Eq. (4). The solution of this equation, satisfying the boundary condition, may be written as

$$p = \int_{-\infty}^{\infty} \psi(k_x) \exp\left[iky\left(1 - \frac{k_x^2}{2k^2}\right) - ik_x x\right] \mathrm{d}k_x \tag{8}$$

where

$$\psi(k_x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p_0(x) \exp(ik_x x) \mathrm{d}x \tag{9}$$

In the following the explicit dependence of ψ on k_x will often be suppressed. Higher order approximations can be obtained from Eq. (6) by neglecting terms of $\mathcal{O}(1/k^n)$ for any value of n. However, these approximations contain derivatives of the unknown function $A(x,y)$ with respect to y. Use of the fact that A_y is of $\mathcal{O}(1/k)$ allows systematic replacement of these terms with derivatives of A with respect to x. This process will be illustrated by obtaining an $\mathcal{O}(1/k^3)$ approximation. To this order, Eq. (6) is

$$A_y \simeq (i/2k)A_{xx} + (i/2k)^2 A_{yxx} + (i/2k)^3 A_{yyxx} \tag{10}$$

The second and third terms on the right-hand side of this equation are multiplied by $1/k^2$ and $1/k^3$, respectively. Remembering that $\mathcal{O}(1/k^4)$ terms have already been neglected in this equation, it is clear that use of an $\mathcal{O}(1/k^2)$ approximation for A_{yxx} and an $\mathcal{O}(1/k)$ approximation for A_{yyxx} in Eq. (10) will yield an equation that is asymptotically equivalent to the equation in its present form. The fundamental assumption of the method is that A_y is $\mathcal{O}(1/k)$, while derivatives of $A(x;k)$ with respect to x are $\mathcal{O}(1)$. Thus A_{yyxx} is $\mathcal{O}(1/k)$ and the last term of Eq. (10) is negligible in its entirety. Equation (10), itself, implies that $A_y = (i/2k)A_{xx} + \mathcal{O}(1/k^2)$. Differentiating this asymptotic equality twice with respect to x, using the resulting expression to eliminate A_{yxx}, and neglecting terms explicitly proportional to $(1/k^4)$ and higher, yields

$$A_y = (i/2k)A_{xx} + (i/2k)^3 A_{xxxx} \tag{11}$$

Note that this equation is fourth-order and that the $\mathcal{O}(1/k^2)$ approximation in this sequence is the same as the $\mathcal{O}(1/k)$ approximation. Higher order approximations will contain higher order derivatives of $A(x,y)$, although these will always be derivatives in directions transverse to the rays—that is, derivatives in the x direction in the current example. Further, for inhomogeneous media, an $\mathcal{O}(1/k^2)$ approximation that is different than the $\mathcal{O}(1/k)$ approximation does appear in the sequence.

The solution of Eq. (11) is

$$p = \int_{-\infty}^{\infty} \psi \exp\left[iky\left(1 - \frac{k_x^2}{2k^2} - \frac{k_x^4}{8k^4}\right) - ik_x x\right] \mathrm{d}k_x \tag{12}$$

while the solution of Eq. (1) may be written as

$$p = \int_{-\infty}^{\infty} \psi \exp\left[iky\left(1 - \frac{k_x^2}{k^2}\right)^{1/2} - ik_x x\right] \mathrm{d}k_x \tag{13}$$

Expansion of the radical in Eq. (13) for $k_x/k < 1$ yields Eqs. (7), (8), and (12), successively. Higher order approximations of this radical will provide solutions of Eq. (6) after this equation is further approximated, as illustrated above. If the angle ϕ defined by

$$\cos\phi = k_y/k \tag{14a}$$

$$\sin\phi = k_x/k \tag{14b}$$

is introduced, it becomes clear that $(1 - k_x^2/k^2)^{1/2} = (1 - \sin^2\phi)^{1/2}$ is being approximated for small values of $\sin\phi$ with each succeeding equation in the sequence providing one more term of the expansion. This is why parabolic approximations are generally referred to as small-angle approximations. Traditionally, $\sin^2\phi$ is also expanded for small values of ϕ; for the zeroth (geometric acoustics), and first-order approximations this causes no difference in the resulting equations. In the current work, these approximations are considered high-frequency, or large wave number approximations. These two interpretations are consistent if the source wave number spectrum is assumed fixed. In the preceding example, this implies that $\psi(k_x)$ is fixed as $k \to \infty$. Then, for a given value of k_x, Eq. (14b) implies that $\sin\phi \to 0$ as $k \to \infty$. This is in keeping with the fundamental philosophy of asymptotic analysis even though one would expect the complete description of the source field to require more and, hence, larger, k_x values as $k \to \infty$.

In the next section the procedure outlined here will be applied to obtain a similar sequence of approximations to the equations governing the propagation of acoustic disturbances in a stratified moving medium.

Theoretical Development

Consider a y-stratified inviscid nonconducting ideal gas in steady motion along the positive x direction at velocity $U(y)$. The total fluid pressure, density, and velocity of the medium are written as $P+p$, $\rho(y)+\delta$, and $[U(y)+u,v,w]$, respectively. Here p, δ, and $\boldsymbol{u}=(u,v,w)$ are infinitesimal disturbances of the basic steady flow. If the effects of gravity are ignored, P must be a constant and the following linearized equations of motion

$$\frac{D_0\delta}{Dt}+\rho\nabla\cdot\boldsymbol{u}+\rho' v=0 \tag{15}$$

$$\rho\frac{D_0 u}{Dt}+\rho U' v+p_x=0 \tag{16}$$

$$\rho\frac{D_0 v}{Dt}+p_y=0 \tag{17}$$

$$\rho\frac{D_0 w}{Dt}+p_z=0 \tag{18}$$

$$\frac{D_0 p}{Dt}-c^2\left(\frac{D_0\delta}{Dt}+\rho' v\right)=0 \tag{19}$$

are easily derived from the principles of conservation of mass, linear momentum, and energy. In these equations D_0/Dt is the linearized material derivative operator, $\partial/\partial t+U(y)\partial/\partial x$, and $c^2(y)=\gamma P/\rho$ is the speed of sound within the undisturbed medium. The prime notation indicates ordinary differentiation of the mean state quantities with respect to y, while the subscript notation indicates partial differentiation of the acoustic quantities with respect to the indicated variable. It should be noted at the outset that the following analysis could be carried out even if the gravitational force were not neglected. However, maintaining these effects causes an increased complexity that was considered unnecessary for the purposes of the current study. Further, Eqs. (15-19) are in nondimensional form, the relevant mass, length, and time scales are $\rho_0\ell^3$, ℓ, and ℓ/c_0, respectively, ρ_0 and c_0 are the mean density and sound speed at, say, $y=0$, while ℓ is the smallest length scale pertinent to the problem of interest. Thus, ℓ may be related to the source size, or it may represent the length scale over which the medium's sound speed or mean velocity varies. In any case, for the theory to be developed subsequently, the wavelength, $\lambda^*=2\pi/k^*$, of the acoustic disturbance is assumed small when compared with ℓ. Here λ^* and k^* are the dimensional wavelength and wave number.

If the dependent variables u, w, and δ are eliminated from the preceding system of equations,

$$\frac{D_0^2 p}{Dt^2}-c^2\nabla^2 p-2U' v_x-2cc' p_y=0 \tag{20}$$

and Eq. (17) are obtained for the unknown functions p and v. It is this system of equations that is used in the analysis to follow.

Consideration is given to the acoustic field maintained in the stratified moving medium due to some arbitrary bounded source distribution. It is assumed that the acoustic pressure is known on some surface $S(x)=0$ as a function of position on that surface, and that a radiation condition is to be imposed. Also the inhomogeneities of the medium are assumed restricted to some region bounded in y. Then solutions of Eqs. (17) and (20) of the form

$$p=A(x;k)\psi(x)e^{ik(\theta-t)} \tag{21}$$

$$v=\nu(x;k)e^{ik(\theta-t)} \tag{22}$$

are sought. Here the wave number k and circular frequency are equal due to the nondimensionalization. Substitution of Eqs. (21) and (22) into Eqs. (20) and (17), respectively, yields

$$\begin{aligned}&(ik)^2(|\nabla\theta|^2-q^2)A\psi+(ik)A\{2(\nabla\psi\cdot\nabla\theta+Mq\psi_x)\\&+[\nabla^2\theta-M^2\theta_{xx}-(2q'\theta_y/q)]\psi\}\\&+2ik\psi(\nabla A\cdot\nabla\theta+MqA_x)+(2N^3\nu_x/q)(M/N)'\\&-2(q'/q)(A\psi_y+A_y\psi)+(\nabla^2\psi-M^2\psi_{xx})A\\&+2(\nabla A\cdot\nabla\psi-M^2A_x\psi_x)+(\nabla^2A-M^2A_{xx})\psi=0\end{aligned} \tag{23}$$

$$\nu=\frac{A\psi\theta_y}{Nq}-\frac{iM\nu_x}{kq}-\frac{i(A\psi_y+A_y\psi)}{Nkq} \tag{24}$$

where

$$q=N-M\theta_x \tag{25}$$

$$q'=N'-M'\theta_x \tag{26}$$

$$N=1/c \tag{27}$$

$$M=U/c \tag{28}$$

and use has been made of Eq. (24) to eliminate a term proportional to $(ik\nu)$ that would have otherwise appeared in Eq. (23). In the following it is assumed that $|M|<1$.

The functions θ and ψ are now required to satisfy the equations

$$|\nabla\theta|^2=q^2 \tag{29}$$

$$2(\nabla\psi\cdot\nabla\theta+Mq\psi_x)+[\nabla^2\theta-M^2\theta_{xx}-2q'\theta_y/q]\psi=0 \tag{30}$$

which are recognized as the eiconal and lowest order transport equations of the geometric acoustics theory. In the lowest order geometric theory, the surfaces $\theta=$const are surfaces of constant phase. In the theory to be presented here, however, as in higher order geometric acoustics expansions, the surfaces of constant phase are no longer represented by $\theta=$const, although any change from these surfaces is $\mathcal{O}(1/k)$ as $k\to\infty$. The quantity q in Eq. (29) may be thought of as a refractive index for the medium, although here its value depends upon the direction of propagation as well as the location within the medium[7] [see Eq. (25)]. Equation (30) may be discussed in several equivalent ways. Traditionally, this equation is used to show that

$$\nabla\cdot\boldsymbol{E}=\nabla\cdot[\psi^2(1+2M\hat{\theta}\cdot\hat{x}+M^2)^{1/2}\hat{\alpha}/q]=0 \tag{31}$$

where $\hat{\alpha}=(\hat{\theta}+M\hat{x})/(1+2M\theta_x/|\nabla\theta|+M^2)^{1/2}$ is a unit vector along the geometric acoustics ray and $\hat{\theta}=\nabla\theta/|\nabla\theta|$ is a unit vector normal to a surface given by $\theta=$const. The vector $\boldsymbol{E}$ is, of course, the geometric acoustics energy flux vector. Letting A_r be the local cross-sectional area of a ray tube, Eq. (31) implies that $|\boldsymbol{E}|A_r$ is constant along the ray tube.

In the geometric theory it is consistent to interpret $\boldsymbol{E}$ as the acoustic energy flux vector, at least to lowest order. However, there are situations wherein the acoustic energy is not constrained to remain within the ray tube (a shadow boundary is a prominent example); here the geometric theory breaks down due to the neglect of diffraction effects. Further, in the theory to be developed herein, the acoustic pressure is proportional to the as-yet undetermined function $A(x,k)$ and the vector E being independent of $A(x,k)$ cannot reasonably be interpreted as an energy flux vector. Since the function $A(x;k)$ describes the diffraction effects neglected in the geometric theory, this function accounts for the loss or gain of acoustic energy through the side walls of the ray tube. However, it should be noted that for those cases where the lowest order geometric acoustics approximation provides an exact solution of Eqs. (25-30), the function $A(x,k)$ is a constant. Here then, the traditional interpretation of Eq. (30) remains valid. Further note that higher order geometric acoustics approximations will not constrain the acoustic energy to the ray tube; these higher order corrections acount for the same effects as the function $A(x)$. However, the geometric acoustics expansion is generally secular in the variable θ.[11]

Equation (29) is to be solved subject to the conditions that $\theta=\theta_0(x)$ on the surface $S(x)=0$, and that θ satisfies an outgoing condition.[12] Use of Eqs. (29) and (30) in Eq. (23) yields the equation

$$2ik\psi(\nabla A\cdot\nabla\theta+MqA_x)+\frac{2N^3}{q}\left(\frac{M}{N}\right)'\nu_x-\frac{2q'(A\psi_y+A_y\psi)}{q}+(\nabla^2\psi-M^2\psi_{xx})A+2(\nabla A\cdot\nabla\psi-M^2A_x\psi_x)+(\nabla^2A-M^2A_{xx})\psi=0 \qquad (32)$$

for the function $A(x;k)$. The first term in this equation is proportional to $\nabla A\cdot\hat{\alpha}$, where $\hat{\alpha}$ is the unit vector along the geometric acoustics ray direction as previously defined. This term is multiplied by k and, as in the case for the homogeneous medium discussed previously, as $k\to\infty$ this term will be balanced by different terms of Eq. (32) depending on whether $\nabla A\cdot\hat{\alpha}\sim\mathcal{O}(k)$ or $\nabla A\cdot\hat{\alpha}\sim\mathcal{O}(1/k)$. Equation (32) is not in a convenient form to determine which terms are of the same order as $\nabla A\cdot\hat{\alpha}$. Introduction of new independent variables—θ, ξ, and η—into Eq. (32) will place this equation in a form that will simplify identification of those terms that must be maintained. Equations which ξ and η must satisfy will be determined in the course of the analysis, although it is to be anticipated that coordinates normal to the ray are what is sought.

Performing the indicated change of variable places Eq. (32) in the form

$$2ik\left(A_\theta+\frac{\nabla\xi\cdot\nabla\theta+Mq\xi_x}{Nq}A_\xi+\frac{\nabla\eta\cdot\nabla\theta+Mq\eta_x}{Nq}A_\eta\right)+A_{\theta\theta}=-L_1[A]-L_2[\nu]-2\frac{[(\nabla\xi\cdot\nabla\eta)-M^2\xi_x\eta_x]}{Nq}A_{\xi\eta} \qquad (33)$$

where

$$NqL_1[A]=[\,|\nabla\xi|^2-(M\xi_x)^2]A_{\xi\xi}+[\,|\nabla\eta|^2-(M\eta_x)^2]A_{\eta\eta}+g(\theta)A_\theta+g(\xi)A_\xi+g(\eta)A_\eta-2MN[\xi_xA_{\theta\xi}+\eta_xA_{\theta\eta}+\theta_xA_{\theta\theta}/2]+hA \qquad (34)$$

$$L_2[\nu]=\frac{2N^2}{\psi q^2}\left(\frac{M}{N}\right)'[\theta_x\nu_\theta+\xi_x\nu_\xi+\eta_x\nu_\eta] \qquad (35)$$

and

$$g(\beta)=\nabla^2\beta-M^2\beta_{xx}+2\left(\frac{\nabla\psi\cdot\nabla\beta-M^2\psi_x\beta_x}{\psi}-\frac{q'\beta_y}{q}\right) \qquad (36)$$

$$h=\frac{\nabla^2\psi-M^2\psi_{xx}}{\psi}-\frac{2q'\psi_y}{q\psi} \qquad (37)$$

Equation (33) suggests that ξ and η be defined as solutions of the equations

$$\nabla\xi\cdot\nabla\theta+Mq\xi_x=0 \qquad (38)$$

$$\nabla\eta\cdot\nabla\theta+Mq\eta_x=0 \qquad (39)$$

$$\nabla\xi\cdot\nabla\eta-M^2\xi_x\eta_x=0 \qquad (40)$$

thus making A_θ the only term in Eq. (33) multiplied by k.

The first two conditions of Eqs. (38-40) constitute the requirement that $\xi(x)$ and $\eta(x)$ be coordinates normal to the ray. The third condition is to some extent arbitrary and will be discussed later. For the present it is sufficient to note that use of Eq. (40) is convenient because it eliminates the A_{ξ_η} term from Eq. (33). Further observe that these definitions of ξ and η were anticipated in the definition of the operator $L_1[\ ]$, because use of Eqs. (38) and (39) is required to write this operator in the form given in Eq. (34). Use of Eqs. (38-40) in Eq. (33) provides the equality

$$2ikA_\theta+A_{\theta\theta}=-L_1[A]-L_2[\nu] \qquad (41)$$

while performing the change of independent variables in Eq. (24) shows that

$$\nu=(\psi\theta_y/Nq)A+(M/ikq)(\nu_\theta\theta_x+\nu_\xi\xi_x+\nu_\eta\eta_x)+(\psi_y/Nikq)A+(\psi/Nikq)(A_\theta\theta_y+A_\xi\xi_y+A_\eta\eta_y) \qquad (42)$$

Assuming that $L_1[A]$ and $L_2[\nu]$ are $\mathcal{O}(k^0)$, Eq. (41) implies that either $A_\theta\sim\mathcal{O}(k)$ or $A_\theta\sim\mathcal{O}(1/k)$. The first possibility is rejected because it implies that $A\sim\exp(-2ik\theta)$ which forces a solution for p that cannot satisfy the radiation condition that must be imposed for $\theta\to+\infty$. Hence, $A_\theta\sim\mathcal{O}(1/k)$. As in the example presented in the previous section, Eq. (41) is multiplied by the factor $\exp(2ik\theta)$ and integrated from i_∞ to θ to yield

$$A_\theta=-\int_{i\infty}^{\theta}\{L_1[A]+L_2[\nu]\}e^{2ik(\phi-\theta)}\mathrm{d}\phi \qquad (43)$$

or

$$A_\theta=\sum_{j=0}^{\infty}\left(\frac{i}{2k}\right)^{j+1}\frac{\partial^j}{\partial\theta^j}\{L_1[A]+L_2[\nu]\} \qquad (44)$$

after repeated integration by parts. Equation (44) and the fact that $A_\theta\sim\mathcal{O}(1/k)$, while $\nu\sim A+\mathcal{O}(1/k)$ are sufficient to produce, to any order in $(1/k^n)$, an equation for A that contains only the first-order derivative of A with respect to θ. The procedure is similar to, but more complicated than, that illustrated previously. For example, Eqs. (44) and (42) become

$$A_\theta=0 \qquad (45a)$$

$$\nu=A\psi\theta_y/Nq \qquad (45b)$$

respectively, when $\mathcal{O}(1/k)$ terms are neglected. Equation (45a) implies that A is an arbitrary function of ξ and η, thus $A=p_0(\xi,\eta)/\psi|_{S=0}$ where $p_0(\xi,\eta)$ is the acoustic pressure on the surface $S=0$. To this order of approximation, the procedure reproduces the lowest order geometric acoustics theory.

At the next order of approximation, Eq. (44) yields

$$2ikA_\theta = -L_1[A] - L_2[\nu] \tag{46}$$

Let

$$L_3[A] = L_1[A] + L_4[A] \tag{47}$$

where

$$L_4[A] = \frac{M}{q}[2(\xi_x A_{\theta\xi} + \eta_x A_{\theta\xi}) + \theta_x A_{\theta\theta}] - \frac{g(\theta)A_\theta}{Nq} \tag{48}$$

The operator $L_3[\ ]$ is elliptic in the (ξ,η) variables and contains no derivatives of A with respect to θ. The expression $L_4[A]$ is of order $1/k$ since it contains only derivatives of $A(\theta,\xi,\eta;k)$ with respect to θ and the cross derivative terms $A_{\theta\xi}$ and $A_{\theta\eta}$. It is worth noting at this point that the operator $L_4[\ ]$ is proportional to the Mach number; the first term of the operator is explicitly so, while use of the transport equation (30) will shown that $g(\theta)$ is also directly proportional to M. Thus, as $M\rightarrow 0$, the operator $L_3[\ ]\rightarrow L_1[\ ]$.

Since terms of $\mathcal{O}(1/k)$ have already been neglected in Eq. (46) it is consistent to replace the operator $L_1[\ ]$ with the operator $L_3[\ ]$. For higher order approximation this direct replacement is no longer possible, although $L_1[\ ]$ will, to any given order of approximation, be equivalent to an operator that does not contain derivatives with respect to θ. Further, let

$$L_5[\ ] = L_2[\ ] - \frac{2N^2\theta_x}{\psi q^2}\left(\frac{M}{N}\right)' \frac{\partial}{\partial\theta} \tag{49}$$

To leading order $\nu \sim A$ [see Eq. (24)] and, since $A_\theta \sim \mathcal{O}(1/k)$, the expression $L_2[\nu]$ in Eq. (46) may be replaced by

$$L_6[A] = L_5\left[\frac{A\psi\theta_y}{Nq}\right] + \frac{2N^2\theta_x}{\psi q^2}\left(\frac{M}{N}\right)' \frac{\partial}{\partial\theta}\left(\frac{\psi\theta_y}{Nq}\right)A \tag{50}$$

Thus, to this order,

$$2ikA_\theta + L_3[A] + L_6[A] = 0 \tag{51}$$

a parabolic equation for A since neither $L_3[\ ]$ nor $L_6[\ ]$ contains derivatives with respect to θ.

Higher order approximations for $A(\theta,\xi,\eta;k)$ may be obtained in a similar manner. For the general case considered in this section the algebra is rather lengthy and involved, thus not suitable for providing an understanding of the method. For this reason, the analysis is specialized to a more simple case in the next section. Before doing this, however, it is convenient to discuss the meaning of Eqs. (38-40).

Equation (40) was imposed to eliminate the mixed derivative term $A_{\xi\eta}$ from Eq. (33). If the unit vectors

$$\hat{\xi} = \nabla\xi / |\nabla\xi| \tag{52}$$

$$\hat{\eta} = \eta / |\nabla\eta| \tag{53}$$

are introduced, Eqs. (38-40) may be written as

$$\hat{\xi}\cdot\hat{\alpha} = 0 \tag{54}$$

$$\hat{\eta}\cdot\hat{\alpha} = 0 \tag{55}$$

$$\hat{\xi}\cdot\hat{\eta} = M^2(\hat{\xi}\cdot\hat{x})(\hat{\eta}\cdot\hat{x}) \tag{56}$$

respectively. Clearly Eqs. (54) and (55) constrain ξ and η to be coordinates normal to the ray. Relation (56) then serves only to constrain the angle between these coordinates in a plane normal to the ray.

Plane Waves in a Stratified Moving Medium

In the preceding section, a rather general method for deriving higher order parabolic approximations that describe the propagation of acoustic disturbances in a stratified moving medium has been developed. In this section consideration is limited to plane waves propagating with the wave front normal to a small angle ϕ to the positive y axis. Further, the index of refraction $N(y)$ will be assumed equal to unity throughout the medium and the acoustic field will be assumed independent of z; then, if θ is taken as y, Eq. (41) becomes

$$2ikA_y + A_{yy} + A_{\xi\xi} - M'A_\xi - 2MA_{y\xi} + 2M'\nu_\xi = 0 \tag{57}$$

where

$$\xi = x - \int_0^y M(\zeta)\mathrm{d}\zeta \tag{58}$$

If interest is in waves for which the angle of the wave front normal is not at a small angle to the y axis, a larger number of terms of the expansion must be used to describe the field to a given degree of accuracy. Alternatively, $\theta(x)$ may be chosen so that the lowest order approximation more nearly describes the field. This will be discussed further at a later time. Since plane waves are being considered, the transport function ψ can be set equal to a constant, $\psi = 1$, then Eq. (24) for ν becomes

$$\nu + (iM/k)\nu_\xi = A + (iM/k)A_\xi + A_y/ik \tag{59}$$

For the purposes of the current discussion, it is convenient to Fourier transform Eqs. (57) and (59) with respect to ξ. This simplifies the analysis [since differentiation of the unknown functions $A(y,\xi;k)$ and $\nu(y,\xi;k)$ with respect to ξ is replaced by multiplication by the factor $i\mu$, where μ is the transform parameter], but does not alter it in any essential way. Let T and Λ be the Fourier transforms of A and ν, respectively. Fourier transforming Eqs. (57) and (59) and eliminating Λ between the resulting equations yield

$$2ik\mathrm{T}' + \mathrm{T}'' = (\mu^2 - M'i\mu)\mathrm{T} + 2i\mu\left[M + \frac{iM'}{k - M\mu}\right]\mathrm{T}' \tag{60}$$

As before, Eq. (60) is multiplied by the integrating factor $\exp(2iky)$ and integrated. The right-hand side of the resulting equation is then repeatedly integrated by parts to obtain

$$\mathrm{T}' = \sum_{j=0}^{\infty}\left(\frac{i}{2k}\right)^{j+1}\frac{\mathrm{d}^j}{\mathrm{d}y^j}[i\mu(M'\mathrm{T} - 2M\mathrm{T}') - \mu^2\mathrm{T}] + \frac{2\mu}{k}\sum_{j=0}^{\infty}\left(\frac{i}{2k}\right)^{j+1}\frac{\mathrm{d}^j}{\mathrm{d}y^j}\left[M'\sum_{\ell=0}^{\infty}\left(\frac{M\mu}{k}\right)^\ell\mathrm{T}'\right] \tag{61}$$

where it has been assumed that $M\mu < k$ and the series expansion

$$\left(1 - \frac{M\mu}{k}\right)^{-1} = \sum_{\ell=0}^{\infty}\left(\frac{M\mu}{k}\right)^\ell \tag{62}$$

has been used to obtain Eq. (61).

This assumption alone invalidates the approximation in the critical layer where $M\mu = k$; note that when $k/\mu = M$, the phase vleocity of the wave is equal to the flow velocity and it is well known that effects neglected in the governing equations [Eqs. (15-19)] become important.[13,14] Thus, the fact that the approximation is invalid within the critical layer is not considered overly restrictive.

For the purposes of the current discussion, a major advantage of Eq. (61) over the more comprehensive equation (44) is that higher order approximations can be developed more clearly from Eq. (61) than from Eq. (44). Essentially the same procedure can be applied to Eq. (44), however.

If $\mathcal{O}(1/k)$ terms are neglected from Eq. (61), the equation

$$T' = 0 \tag{63}$$

is obtained; as before this yields the lowest order geometric acoustics theory. If $\mathcal{O}(1/k^2)$ terms are neglected, Eq. (61) becomes

$$T' = (-i/2k)[\mu^2 T - i\mu(M'T - 2MT')] \tag{64}$$

Although no essential simplification is obtained by neglecting the T' term on the right-hand side of this equation, it must be remembered that terms of $\mathcal{O}(1/k^2)$ have already been dropped, and that $T' \sim \mathcal{O}(1/k)$ for the solution of interest. Thus $T'/2k$ is of $\mathcal{O}(1/k^2)$, therefore negligible to this order of approximation, and Eq. (64) becomes

$$T' = (-\mu/2k)(i\mu + M')T \tag{65}$$

or

$$2ikA_y + A_{\xi\xi} + M'A_\xi = 0 \tag{66}$$

after an inverse transform; note in particular that this equation is parabolic.

If $\mathcal{O}(1/k^3)$ terms are neglected in Eq. (61), the equation

$$T' = -\mu^2\left[\left(\frac{i}{2k}\right)T + \left(\frac{i}{2k}\right)^2 T'\right] + \frac{5i\mu}{4k^2}M'T'$$
$$+ i\mu\left[\left(\frac{i}{2k}\right)(M'T - 2MT') + \left(\frac{i}{2k}\right)^2(M''T - 2MT'')\right] \tag{67}$$

is obtained. The second, third, and seventh terms on the right-hand side of this equation are proportional to $1/k^2$; further, these are differentiated terms that are $\mathcal{O}(1/k)$. Since $\mathcal{O}(1/k^3)$ terms have already been discarded, these terms may be treated similarly. The fifth term on the right-hand side of Eq. (67) is proportional to $1/k$, thus on $\mathcal{O}(1/k^2)$ expression for T' is required. This is obtained from Eq. (65). With the indicated approximations, Eq. (67) becomes

$$T' = (i/2k)(i\mu)(i\mu + M')T$$
$$+ (i/2k)^2[i\mu M'' - 2M(i\mu)^2(i\mu + M')]T \tag{68}$$

or

$$2ikA_y + A_{\xi\xi} + M'A_\xi$$
$$+ (i/2k)[M''A_\xi - 2M(A_{\xi\xi\xi} + M'A_{\xi\xi})] = 0 \tag{69}$$

after the inverse transform.

Higher order approximations may be obtained in a similar fashion. The major steps of the procedure are: 1) neglect terms of $\mathcal{O}(1/k^n)$ from Eq. (61) and 2) use the fact that T' is of $\mathcal{O}(1/k)$ to approximate derivatives of T [derivatives of $A(x;k)$ in the ray direction in the general theory] in a consistent manner. Here use may always be made of previously obtained lower order approximations.

The same procedure is applied in the more general case discussed in the previous section. Here, however, an equation on $A(\theta,\xi,\eta;k)$ alone can only be obtained at each order of approximation by use of an approximate equation for ν. For the case discussed in this section, the series given in Eq. (62) does this part of the analysis automatically. Essentially, the procedure outlined here systematically replaces derivatives of $A(\theta,\xi,\eta;k)$ along the ray with derivatives transverse to the rays. These derivatives are related by the governing differential equations. The procedure provides a mechanical means for maintaining those terms that must be maintained for an $\mathcal{O}(1/k^n)$ approximation for any value of n. For the first-order parabolic approximation the resulting equations are conveniently solved numerically by marching techniques; this is the virtue of the method.

For the higher order approximations derived here, however, this numerical integration would be best carried out by replacing the given approximate equation with a system of first-order equations.[15] This system would then be solved subject to a set of auxiliary conditions on the surface $S(x) = 0$. There is a certain degree of arbitrariness in the application of these conditions since the product $A\psi$ evaluated at $S = 0$ must give the known acoustic pressure there. Generally, $\psi|_{S=0}$ can be given the constant value unity; then $A|_{S=0} = p_0(\xi,\eta)$.

Numerical Example

In order to obtain some insight into the accuracy of the approximations derived in the previous section, consider plane waves incident upon a shear layer separating two regions of homogeneous fluid moving along the x axis at uniform velocities M_0 and M_1, respectively. The thickness of the shear layer will be used as the relevant length scale and the shear layer will be taken as located between $y = 0$ and 1. The mean flowfield is then given by

$$\begin{aligned} M(y) &= M_0, & y &\le 0 \\ &= (M_1 - M_0)f(y) + M_0, & 0 &\le y \le 1 \\ &= M_1, & y &\le 1 \end{aligned} \tag{70a}$$

where

$$f(y) = y^3(6y^2 - 15y + 10) \tag{70b}$$

The approximation for T for a fixed value of μ may be written

$$\ell n T = T_1 + T_2 + T_3 \tag{71}$$

where T_1, T_2, and T_3, given by

$$T_1 = -\frac{\mu}{2k}(i\mu y + M - M_0) \tag{72a}$$

$$T_2 = -\frac{i\mu}{4k^2}\left[M' - i\mu(M^2 - M_0^2) + 2\mu^2\int_0^y M(\zeta)\,d\zeta\right] \tag{72b}$$

$$T_3 = -\frac{\mu^2}{6k^3}[\mu(M^3 - M_0^3) + 3iMM']$$
$$+ \frac{\mu^3}{2k^3}\left[M - M_0 - i\mu\int_0^y M^2(\zeta)\,d\zeta\right]$$
$$-\frac{\mu}{8k^3}\left[i\mu^3 y - M^{(2)} + i\mu\int_0^y [M'(\zeta)]^2\,d\zeta\right] \tag{72c}$$

are $\mathcal{O}(1/k)$, $\mathcal{O}(1/k^2)$, and $\mathcal{O}(1/k^3)$, respectively, for fixed μ. In this example, μ may be identified as the x component of the wave number vector; thus $\mu/k = \sin\phi/(1 - M\sin\phi)$, where ϕ is the angle of the wave front normal to the positive y axis. In the above expressions, terms proportional to μ^2/k, μ^3/k^2, etc., appear for fixed μ and $k \to \infty$. These terms are properly ordered in the preceding expansion. However, if these terms are written in terms of $\sin\phi$ they become $k\sin^2\phi/(1 + M\sin\phi)^2$ and $k^2\sin^3\phi/(1 + M\sin\phi)^3$, respectively, and, for fixed ϕ, the above ordering is incorrect as $k \to \infty$. Thus, the approximation may be properly interpreted as either a high-frequency expansion for a fixed component of the wave number normal to the ray or as a small-angle approximation for fixed frequency. As the approximation is developed in this and the previous section this small angle is to the positive y axis, with $\phi < 0$ for upstream propagation. Use of another solution to the eiconal equation, Eq. (29), would allow consideration of waves propagating at a small angle to a different reference direction.

Solutions of Eqs. (18) and (20) for plane waves incident upon a shear layer can be found rather easily by numerical methods.[16] Comparisons of these numerical solutions with

$$\exp\left(\bar{\mathrm{T}} - i\mu\int_0^y M(\zeta)\mathrm{d}\zeta + iky\right)$$

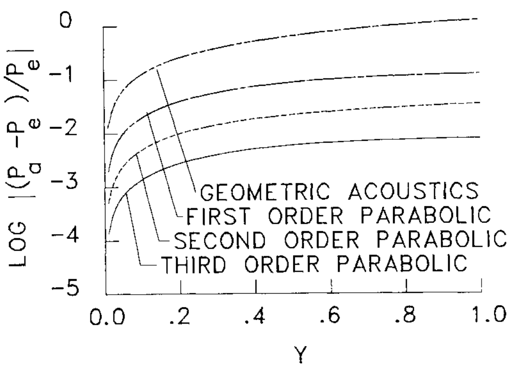
are given in Figs. 1-3. Here the geometric acoustics solution is obtained for $\bar{\mathrm{T}} = 0$, while the first-order parabolic solution is obtained for $\bar{\mathrm{T}} = \mathrm{T}_1$; the second- and third-order parabolic solutions are given when $\bar{\mathrm{T}} = \mathrm{T}_1 + \mathrm{T}_2$ and $\mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3$, respectively. Note that T_1, T_2, and T_3 are as given in Eqs. (72).

In Fig. 1 the quantity $\log_{10}|(P_e - P_a)/P_e|$ is plotted against the vertical (y) position within the shear layer. Here P_e is the exact (numerical) solution and P_a the approximate solution. For the calculations presented here $M_0 = 0.7$, $M_1 = 0.2$, $k = 50$, and the angle of the wave front normal to the y axis ϕ is -12 and $+12$ deg for the results plotted in Figs. 1a and 1b, respectively. Note that negative angles correspond to waves propagating upstream, while positive angles yield waves propagating downstream. For propagation in stationary inhomogeneous media, the standard parabolic approximation is generally considered valid for waves propagating up to $\sim \pm 7$ deg from the reference direction[5] ϕ_r, here $\phi_r = 0$ deg. The accuracy obtained for waves propagating within this range of angles would depend on the value of k; this will be illustrated later. Thus, for the cases shown in Fig. 1 the wave considered is propagating at an angle for which standard parabolic approximations are considered invalid. Indeed the first-order parabolic approximation yields a solution whose distance from the numerical solution in the complex plane is $|P_e|/10$, while the distance of the third-order parabolic approximation from the exact solution is $\sim |P_e|/100$. (Note that the distance from the exact solution to the geometric acoustics solution is $|P_e|$.) See Fig. 1.

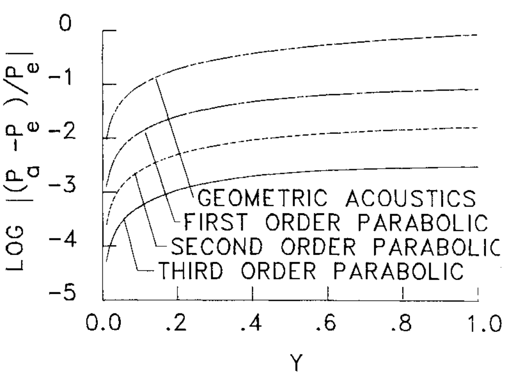
In general, parabolic approximations include an error that increases with propagation distance. This is clearly exhibited in Fig. 1. However, as is to be expected, the higher order approximations are better at all points within the range of interest, $0 \le y \le 1$, except at $y = 0$, where all solutions agree since they are required to satisfy the same auxiliary condition. Solutions were obtained for a wide range of values of k and propagation angles. The results shown here are typical. Figure 2 displays $\log_{10}|(P_a - P_e)/P_e|$ at $y = 1$ as a function of propagation angle ϕ for -10 deg$<\phi<10$ deg. As can be seen, the approximation is better for ϕ near 0 deg and worse as ϕ gets larger in magnitude. This is typical of parabolic approximations. Note that the higher order approximations extend the range of propagation angles over which a given error tolerence can be met.

Figure 3 shows how the accuracy of the approximation for a fixed value of μ improves as k increases. For the calculations considered here μ is unity while k varies from 2.5 to 52.5; the angle ϕ is thereby varied from 33.75 to 1.11 deg. The approximation is seen to improve as k increases.

The examples presented here are intended only to provide insight into the nature of parabolic approximations, espe-



a) $\phi = -12$ deg.



b) $\phi = 12$ deg.

Fig. 1 Comparison of approximate and exact solutions as a function of distance through the shear layer.

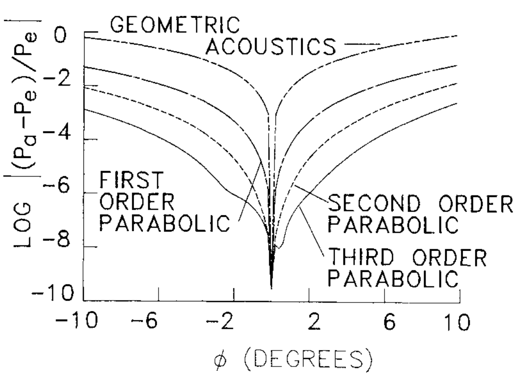


Fig. 2 Comparison of exact and approximate solutions at $z = 1.0$ as a function of propagation angle.

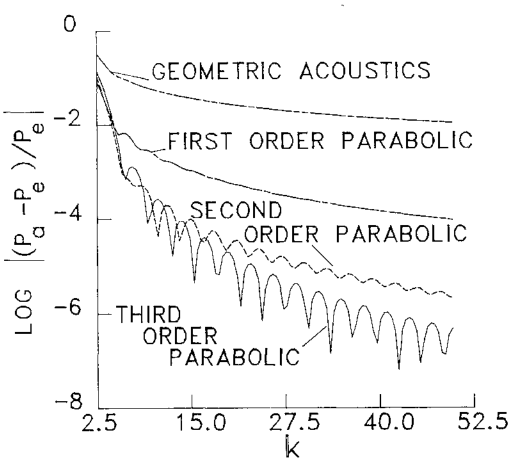


Fig. 3 Comparison of exact and approximate solutions at $z = 1.0$ for fixed μ as a function of k.

cially as these approximations apply to stratified moving media. The particular examples considered in this last section can be solved adequately by Fourier analysis and numerical integration of the resulting second-order ordinary differential equations in most cases. For large values of k, however, this procedure is not as efficient as use of the approximate solution as given in Eqs. (72) which yields a solution adequate for most purposes, at least for wave components propagating nearly perpendicular to the shear layer.

Concluding Remarks

In this paper a method has been developed which provides a parabolic differential equation whose solution provides to $\mathcal{O}(1/k^n)$, the acoustic pressure field or infinitesimal disturbances propagating in a stratified, inhomogeneous, moving medium. Here k is a dimensionless wave number and n the arbitrary order of the approximation. Simple examples have been used both to motivate the analysis and illustrate the nature and accuracy of the approximation. It has been shown that, at least for the examples considered, the higher order equations improve the accuracy of the approximation and extend its region of validity.

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